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Effects of laser spatial incoherence with finite correlation length on the space-time behavior of backscattering instabilities

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The average space-time behavior of backscattering instabilities in the presence of a purely spatial incoherence of the laser beam is investigated analytically. It is found that the divergence of the strongly damped limit of the average backscattered light amplitude [cf. H. A. Rose and D. F. Dubois, Phys. Rev. Lett. **72**, 2883 (1994)] must be replaced by the onset of an absolutelike instability characterized by a *local*, space dependent growth rate. The validity of the results is discussed in the regime where the breakdown of our model is caused by the onset of absolutely unstable hot spots. The time asymptotic behavior of the instability in this regime is computed heuristically. The average Green's function for the backscattered light is computed analytically in the case where the laser incoherence is characterized by an exponentially decaying correlation function.

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The effects of driver incoherence on parametric instabilities have been the topic of many theoretical studies since the middle of the 1970's [1]. So far, all these studies have been devoted to computing thresholds and growth rates with the assumption that the correlation time or length of the turbulence is very short. Considering the case of laser spatial incoherence induced by random phase plate (RPP) optics, Rose and DuBois have recently reconsidered this problem without assuming a short correlation length [2]. Restricting themselves to the strongly damped convective limit, they have shown that the existence of statistically significant intense hot spots leads to the divergence of the average linear amplification factor as the average laser intensity approaches a critical threshold values [3]. A physical interpretation of this result is very questionable, since the strongly damped approximation is clearly not valid in the intense hot spots that are responsible for the divergence. Actually, one can demonstrate from simple scaling arguments on the gain factor in the weakly damped regime that no divergence of the linear amplification factor is to be expected [4]. This shows that a strongly damped model cannot properly describe the instability near the critical intensity. In this paper, we reconsider the problem raised by Rose and DuBois *without* restricting ourselves to the strongly damped regime. We will study the space-time response of the average amplitude of the backscattered light in the limit of a nonpropagative longitudinal wave. We will see that, in this limit, the divergence of the amplification factor must be replaced by the onset of an absolutelike behavior of the instability characterized by a *local*, space dependent, growth rate.

This paper is organized as follows. First, we study the time asymptotic behavior of the average linear response of the backscattered light. Next, we heuristically demonstrate that an ensemble of independent hot spots leads to the onset of an absolute behavior when the average convective gain length is smaller than the hot spot length. Then we determine the validity domain of our results in the regime where the breakdown of our model is caused by the onset of absolutely unstable hot spots and we heuristically compute the time asymptotic behavior of the instability in this regime. Finally, we perform the *exact analytical* computation of the average Green's function for the backscattered light in the case where the laser incoherence is characterized by an exponentially decaying correlation function.

We consider the one-dimensional linear space-time evolution of the backscattered light (wave 1) in the limit where the envelope approximation is valid for this wave and where the propagation of the longitudinal wave (wave 2) can be neglected. We do not make any envelope approximation for the longitudinal wave, so that our results apply to both the so-called weak and the strong coupling regimes. We neglect the pump depletion, but we keep the spatial variations of the amplitude of the coupling parameter when they result from the incoherence of the pump itself. We study the space-time behavior of the backscattered light by means of the Green's functions $G_{1\beta}$ defined as the solutions to the system,

$$(\partial_t+V_1\partial_x+\nu_1)G_{1\beta}-\gamma_0 S(x)G_{2\beta}^*=\delta_{\beta,1}\delta(x)\delta(t), \quad (1a)$$

$$\left(\frac{i}{2\omega_2}(\partial_{t^2}^2+2\nu_2\partial_t)+(\partial_t+\nu_2)\right)G_{2\beta}-\gamma_0 S(x)G_{1\beta}^*=\delta_{\beta,2}\delta(x)\delta(t), \quad (1b)$$

where ω_α, ν_α, and V_α denote, respectively, the angular frequency, the linear damping, and the group velocity of wave α, with $V_1>0$ and $V_2=0$. The quantity $\delta_{\beta,\alpha}$ on the right-hand side of Eqs. (1) is the Kronecker function. The coupling constant γ_0 is given by the expression of the homogeneous weak coupling growth rate in the coherent case and the quantity $S(x)$ is a stochastic function normalized to $\langle|S(x)|^2\rangle=1$, where $\langle\ \rangle$ denotes statistical average. In the following, S will be taken as a translationally invariant Gaussian process with $\langle S(x)\rangle=\langle S(x)S(x')\rangle=0$ and $\langle S(x)S^*(x')\rangle=C(x-x')\neq 0$. For each realization of S, System (1) can be easily solved using the Laplace transform. The average linear response $\langle G_{1\beta}\rangle$ of the backscattered light is then obtained by averaging over the realizations of S. One finds

$$\langle G_{11}\rangle=(2\pi|V_1|)^{-1}H(x/V_1)e^{-\nu_1 x/V_1}\times\int_L M(\omega,x)e^{-i\omega(t-x/V_1)}d\omega, \quad (2)$$

where $H(x)$ is the Heaviside step function and $M(\omega,x)$ is the average linear amplification factor for the (complex) angular frequency ω,

$$M(\omega,x)=\left\langle \exp\left[l_G^{-1}P(\omega)^{-1}\int_0^x |S(x')|^2 dx'\right]\right\rangle \tag{3}$$

Here, $l_G \equiv |V_1|\nu_2/\gamma_0^2$ is the convective gain length in the coherent case and $P(\omega)$ is a second order polynomial given by

$$P(\omega)=\frac{i\nu_2}{2\omega_2}\left(\frac{\omega}{\nu_2}\right)^2 - i\left(1-\frac{i\nu_2}{\omega_2}\right)\frac{\omega}{\nu_2}+1. \tag{4}$$

The non diagonal response $\langle G_{12}\rangle$ is found to vanish. This result, first pointed out by Thomson and Karush [5], means that in the case where S is a Gaussian process with $\langle S\rangle=0$, the average amplitudes of the two daughter waves are decoupled. Before performing the detailed computation of the rhs of Eq. (2), it is interesting to study the time asymptotic behavior of $\langle G_{11}\rangle$ at a given point x. This behavior is given by

$$\langle G_{11}\rangle(x,t)\sim \exp[-i\omega_{\max}(x)t], \tag{5}$$

where $\omega_{\max}(x)$ is the singularity of $M(\omega,x)$ that has the greatest imaginary part. Denoting by $\kappa_0^{-1}(x)$ the value of the convective gain length at which $M(0,\ x)$ diverges, one finds that $\omega_{\max}(x)$ is the solution to $P(\omega)=[\kappa_0(x)l_G]^{-1}$:

$$\omega_{\max}(x)=\omega_2 - i\nu_2 - \left(\omega_2^2-\nu_2^2-\frac{2i\omega_2\nu_2}{\kappa_0(x)l_G}\right)^{1/2}. \tag{6}$$

One can see that the sign of $\gamma_{\max}(x)\equiv \mathrm{Im}[\omega_{\max}(x)]$ changes as x passes through the critical value x_{cr} defined by $\kappa_0(x_{\mathrm{cr}})=l_G^{-1}$. This means that there is a change in the nature of the instability at the interface between the slabs $0<x<x_{\mathrm{cr}}$ and $x>x_{\mathrm{cr}}$. For $0<x<x_{\mathrm{cr}}$, one has $\gamma_{\max}(x)<0$ so that $\lim_{t\to+\infty}\langle G_{11}\rangle=0$, which corresponds to either a convective instability or a stable behavior. On the other hand, for $x>x_{\mathrm{cr}}$, one has $\gamma_{\max}(x)>0$ so that $\lim_{t\to+\infty}\langle G_{11}\rangle=\infty$, which corresponds to an absolute instability characterized by the *local* growth rate $\gamma_{\max}(x)$. Such a remarkable behavior can be heuristically explained by considering an ensemble of independent hot spots of length $\Lambda_\parallel$ [2]. According to recent works [6,7] on the space-time behavior of parametric instabilities in the coherent case, one can define three different kinds of hot spots depending on their normalized peak intensity I_{hot}, with $\langle I_{\mathrm{hot}}\rangle=1$. In the convective hot spots defined by $I_{\mathrm{hot}}<(l_G/\Lambda_\parallel)(\nu_2 t)$, the instability convectively saturates and the amplitude a_1 of the backscattered light does not depend on time. In the weakly coupled hot spots defined by $(l_G/\Lambda_\parallel)(\nu_2 t)<I_{\mathrm{hot}}<(\omega_2/\nu_2)^2(l_G/\Lambda_\parallel)(\nu_2 t)$, the instability has not reached its convectively saturated steady state yet, and a_1 behaves as

$$a_1(I_{\mathrm{hot}},t)\sim \exp[2\gamma_0(I_{\mathrm{hot}}\Lambda_\parallel t/V_1)^{1/2}-\nu_2 t]. \tag{7a}$$

Finally, in the strongly coupled hot spots defined by $I_{\mathrm{hot}}>(\omega_2/\nu_2)^2(l_G/\Lambda_\parallel)(\nu_2 t), a_1$ behaves as

$$a_1(I_{\mathrm{hot}},t)\sim \exp[3^{3/2}\omega_2(\gamma_0 t/4\omega_2)^{2/3}(I_{\mathrm{hot}}\Lambda_\parallel/V_1)^{1/3}]. \tag{7b}$$

It is known [8] that for high values of I_{hot} the probability density distribution function $P(I_{\mathrm{hot}})$ satisfies $P(I_{\mathrm{hot}})\sim\exp(-I_{\mathrm{hot}})$ (besides some logarithmic corrections). The hot spot contribution to the average amplitude of the backscattered light is therefore

$$\langle a_1(t)\rangle_{\mathrm{hot}}\sim\int^{+\infty} a_1(I_{\mathrm{hot}},t)\exp(-I_{\mathrm{hot}})dI_{\mathrm{hot}}. \tag{8}$$

Estimating the leading behavior of Eq. (8) in the large gain factor limit, one obtains the following results: if $\Lambda_\parallel/l_G<1$, the growth of $\langle a_1\rangle$ is determined by the bulk of the convective hot spots and $\langle a_1\rangle$ does not depend on time; if $1<\Lambda_\parallel/l_G<\omega_2/\nu_2$, the growth of $\langle a_1\rangle$ is determined by the rare weakly coupled hot spots and one has

$$\langle a_1(t)\rangle\sim\exp[(\Lambda_\parallel/l_G-1)\nu_2 t]; \tag{9a}$$

finally, if $\Lambda_\parallel/l_G>\omega_2/\nu_2$, the growth of $\langle a_1\rangle$ is determined by the even more rare strongly coupled hot spots and one has

$$\langle a_1(t)\rangle\sim\exp[(3^{3/4}/2)(\omega_2\nu_2\Lambda_\parallel/l_G)^{1/2}t]. \tag{9b}$$

These results confirm the existence of a transition from a convective to an absolute (exponentially growing) instability when $\Lambda_\parallel$ passes through a critical value (here $\Lambda_\parallel/l_G=1$). It can be shown that Eq. (9a) yields the correct value of $\gamma_{\max}(\Lambda_\parallel)$ as given by Eq. (6) in the limit of an infinite correlation length of $S(x)$. In the same limit, Eq. (9b) slightly overestimates $\gamma_{\max}(\Lambda_\parallel)$ by a factor of $3^{3/4}/2\approx1.14$. This discrepancy follows from the fact that the large gain factor limit of the average response [Eq. (5)] is generally different from the average large gain factor limit of the response [Eqs. (7) and (8)].

It is important to notice that Eq. (6) is quite general and is not related to a specific choice of the stochastic process $S(x)$. For instance, the strategy to compute the growth rate of the instability at a given point in the interaction region could be as follows. First, one determines numerically the critical curve at which $M(0,\ x)$ diverges in the plan $(l_G^{-1},\ x)$, as done in Ref. [2]. Then, for a given point x, one determines the quantity $\kappa_0(x)$, as shown in Fig. 1. Finally, inserting the value of $\kappa_0(x)$ in Eq. (6), one obtains the growth rate of the instability at the point x.

All the results discussed previously have been obtained in the limits of both spatial envelope approximation and nonpropagative longitudinal wave approximation ($V_2=0$). The general problem of the validity of these approximations is very intricate and still unsolved. In this paragraph, we will restrict ourselves to the particular case where the validity domain of our results is constrained by the onset of absolutely unstable independent hot spots in which a_1 behaves as

$$a_1(I_{\mathrm{hot}},t)\sim\exp[(2\gamma_0 I_{\mathrm{hot}}^{1/2}|V_2/V_1|^{1/2}-\nu_2)t]. \tag{10}$$

It can be shown *a posteriori* that this regime corresponds to the limit

$$\min[K_b^2,\ K_b(\Lambda_\parallel\omega_2/|V_2|)^{1/2}]>1, \tag{11}$$

where $K_b\equiv\gamma_0\Lambda_\parallel/|V_1V_2|^{1/2}$ is the so-called Kubo number, $\Lambda_\parallel\sim\min(l_c,\ L)$ is the hot spot length, and L is the interaction length. Estimating as previously the contribution of these hot

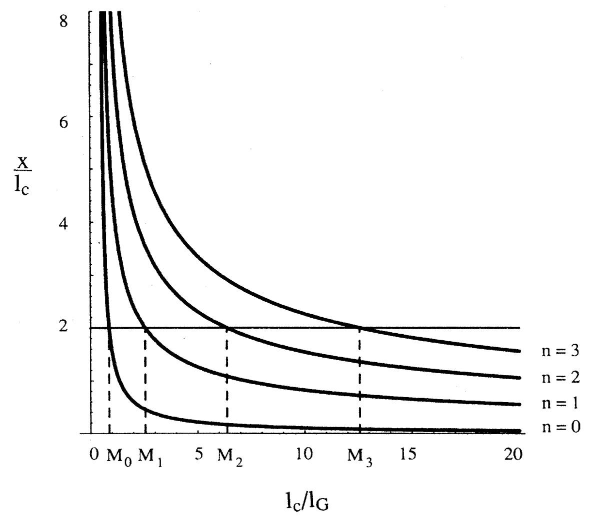

FIG. 1. Critical curves in the plan (l_c/l_G, x/l_c) as given by Eq. (16) (in which κ_n is replaced by l_G^{-1}) for $n=0,1,2,3$. This figure shows how to determine the quantities $\kappa_n(x)$ graphically for a given value of x: here, $\kappa_n(x=2l_c)=M_n/l_c$.

spots to the average amplitude of the backscattered light as the leading behavior of Eq. (8) in the large gain factor limit, one finds

$$\langle a_1(t)\rangle \sim \exp(\gamma_0^2|V_2/V_1|t^2-\nu_2 t). \tag{12}$$

The time $t_{\rm abs}$ at which our model fails can then be estimated by demanding that the gain factor be a continuous function of time at the transition to the regime dominated by the absolutely unstable hot spots. In the regime $\Lambda_\parallel/l_G<1$ where $\langle a_1\rangle$ does not depend on time, one finds that there is no temporal limitation to the validity of our results. In the opposite case, $\Lambda_\parallel/l_G>1$, Eqs. (9a), (9b), and (12) yield the condition

$$t<t_{\rm abs}\equiv(\Lambda_\parallel/|V_2|)\min[1,\sqrt{(l_G/\Lambda_\parallel)(\omega_2/\nu_2)}], \tag{13}$$

where we have ignored unimportant numerical factors of order unity. In the regime $1<\Lambda_\parallel/l_G<\omega_2/\nu_2$, the growth of $\langle a_1\rangle$ is determined by the weakly coupled hot spots and one has $t_{\rm abs}=\Lambda_\parallel/|V_2|$. At this time, the propagation of the longitudinal wave starts playing a role and the approximation

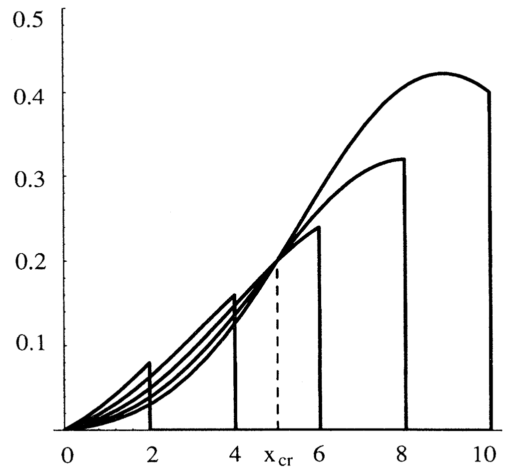

FIG. 2. Evolution of $\mathscr{S}_-$ as given by Eq. (18) in the weak coupling limit (i.e., in the limit $\omega_2\to+\infty$) with $\nu_1=0$, $\nu_2=1$, $l_G=5$, $V_1=5$ for the successive times $t=0.4$, 0.8, 1.2, 1.6, and 2.

$V_2=0$ fails. In the opposite regime, $\Lambda_\parallel/l_G>\omega_2/\nu_2$, the growth of $\langle a_1\rangle$ is determined by the strongly coupled hot spots and one has $t_{\rm abs}=|V_2|^{-1}(l_G\Lambda_\parallel\omega_2/\nu_2)^{1/2}$. At this time, the spatial envelope approximation fails and the anti-Stokes component of the scattered light can no longer be neglected [7]. The validity condition (11) of this scenario is obtained by demanding that Eq. (10) be valid for the leading hot spots (with $I_{\rm hot}=\gamma_0^2|V_2/V_1|t^2$) at the transition time $t_{\rm abs}$.

For the sake of completeness, let us mention that the probability density distribution function $P(I_{\rm hot})$ of a real RPP field must have an upper cutoff at some maximum intensity $I_{\rm max}$. In this case, the actual time asymptotic behavior of $\langle a_1\rangle$ is the same as that of the coherent case [6,7] with the normalized intensity $I_{\rm max}$. The time at which this time asymptotic behavior takes over from the Gaussian RPP field behavior is again obtained by demanding that the gain factor be a continuous function of time.

We now perform the *exact analytical* computation of $\langle G_{11}\rangle$ assuming that the correlation function $C(x)$ is given by $C(x)=\exp(-|x|/l_c)$. In this case, the analytical expression of $M(\omega,x)$ is obtained by substituting $l_G P(\omega)$ for l_G in that of $M(0,x)$ derived in Ref. [4]. Considering the domain of interest where $M(\omega,x)$ may diverge, one finds

$$M(\omega,x)=\frac{2\exp(x/l_c)}{2\cos[\sigma(\omega)x/l_c]+[\sigma(\omega)^{-1}-\sigma(\omega)]\sin[\sigma(\omega)x/l_c]}, \tag{14}$$

where $\sigma(\omega)=[2l_c/l_GP(\omega)-1]^{1/2}$. The singularities of $M(\omega,x)$ as given by Eq. (14) are simple poles located at the angular frequencies

$$\omega_{n\pm}(x)=\omega_2-i\nu_2\pm[\omega_2^2-\nu_2^2-2i(\omega_2\nu_2)/(\kappa_n(x)l_G)]^{1/2}, \tag{15}$$

where $\kappa_n(x)$ is obtained by inverting the relation

$$\frac{x}{l_c}=\frac{1}{\sqrt{2\kappa_n l_c-1}}\left[\tan^{-1}\left(\frac{\sqrt{2\kappa_n l_c-1}}{\kappa_n l_c-1}\right)+n\pi\right], \tag{16}$$

where the determination of $\tan^{-1}$ is such that $0<\tan^{-1}\leq\pi$, and n is a non-negative integer. Equation (16) (in which κ_n is replaced by l_G^{-1}) defines the critical curves displayed in Fig. 1. Inserting these results into Eq. (2) and performing the Laplace integral, one obtains $\langle G_{11}\rangle=g_1^0+\mathscr{G}_-+\mathscr{G}_+$, where $g_1^0(x,t)=H(t)\exp(-\nu_1 t)\delta(x-V_1 t)$ is the uncoupled Green's function and

$$\mathscr{G}_\pm(x,t)=\mp\frac{\gamma_0^2 l_c}{|V_1|^2}H(x(V_1t-x))e^{x(l_c^{-1}-\nu_1/V_1)}\times\sum_{n=0}^{+\infty}g_{n\pm}(x,t), \tag{17a}$$

with

$$g_{n\pm}(x,t)=\frac{(-1)^n}{[\kappa_n(x)l_c]^2}\left[\frac{2\kappa_n(x)l_c-1}{\kappa_n(x)x+1}\right]\frac{\exp[-i\omega_{n\pm}(x)(t-x/V_1)]}{\{1-(\nu_2/\omega_2)^2-2i(\nu_2/\omega_2)[\kappa_n(x)l_G]^{-1}\}^{1/2}}. \tag{17b}$$

The latter expressions exhibit what we will call a *local* normal mode behavior in which each mode "$n\pm$" is characterized by its *local* angular frequency $\omega_{n\pm}(x)$. It can be seen that all the modes "$n+$" are stable so that the contribution of $\mathscr{G}_+$ to $\langle G_{11}\rangle$ is rapidly negligible. On the other hand, $\mathrm{Im}[\omega_{n-}(x)]$ becomes positive as x becomes greater than the value x_n defined by $\kappa_n(x_n)=l_G^{-1}$. Considering then the plan (l_G^{-1},x) displayed in Fig. 1, this means that the mode "$n-$" is locally destabilized when the vertical line corresponding to the actual value of l_G intersects the nth critical curve. If the point (l_G^{-1}, x) is above (respectively below) the critical curve, then the mode is unstable (respectively stable) at this point.

It is interesting to see how $\langle G_{11}\rangle$ reduces in the limit of small or large correlation length l_c. Taking the limit $l_c\to 0$ in Eq. (14) and inserting the result in Eq. (2) one finds that $\langle G_{11}\rangle$ reduces to the coherent Green's function (with $V_2=0$), which is the solution to the Bourret equation for $\langle a_1\rangle$, as it should be [1]. In the opposite limit $l_c\to+\infty$, one has $\kappa_0\approx x^{-1}$ and $\kappa_{n>0}\approx(l_c/2)(n\pi/x)^2$ so that $\mathscr{G}_\pm$ reduces to

$$\lim_{l_c\to+\infty}\mathscr{G}_\pm=\frac{\gamma_0^2 x}{|V_1|^2}H(x(V_1t-x))e^{-\nu_1 x/V_1}\times\frac{\exp[-i\omega_{0\pm}(x)(t-x/V_1)]}{[1-(\nu_2/\omega_2)^2-2i(\nu_2/\omega_2)(x/l_G)]^{1/2}}, \tag{18}$$

which can be obtained directly from Eqs. (1) in which the stochastic function $S(x)$ is replaced by a Gaussian random variable S, with $\langle S\rangle=0$ and $\langle|S|^2\rangle=1$. Figure 2 shows the evolution of $\mathscr{G}_-$ as given by Eq. (18) in the weak coupling limit $\omega_2\to+\infty$ and $\omega_{0-}(x)\approx i\nu_2[(x/l_G)-1]$. One can very clearly see the convective (respectively absolute) nature of the instability for $x<x_{\rm cr}$ (respectively $x>x_{\rm cr}$), with $x_{\rm cr}=l_G$. Namely, at a given *fixed* point $x<x_{\rm cr}$ (respectively $x>x_{\rm cr}$), $\mathscr{G}_-$ is a decreasing (respectively increasing) function of time while its maximum grows as it propagates to the right.

In conclusion, we have analytically computed the average space and time behavior of backscattering instabilities in the case of a purely spatial incoherence of the laser beam. As a result, we have found that the interaction region must be split up into two different domains: The interval $0<x<x_{\rm cr}$ where the instability is convective, and the interval $x_{\rm cr}<x<L$ where the instability is absolute and characterized by a *local*, space dependent growth rate [Eq. (6)]. Considering an ensemble of independent hot spots, we have heuristically determined the validity domain of our results in the regime where our model fails due to the onset of absolute unstable independent hot spots [Eq. (13)], and computed the time behavior of the instability outside this domain [Eq. (12)].

The author would like to thank D. Pesme and H. A. Rose for fruitful discussions he had with them on this topic.

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